

# Growth of graph powers

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## Abstract

For a graph  $G$ , its  $r$ th power is constructed by placing an edge between two vertices if they are within distance  $r$  of each other. In this note we study the amount of edges added to a graph by taking its  $r$ th power. In particular we obtain that either the  $r$ th power is complete or “many” new edges are added. This is an extension of a result obtained by P. Hegarty for cubes of graphs.

## 1 Introduction

This note addresses some questions raised by P. Hegarty in [2]. In that paper he studied results about graphs inspired by the Cauchy-Davenport Theorem.

All graphs in this paper are simple and loopless. For two vertices  $u, v \in V(G)$ , denote the length of the shortest path between them by  $d(u, v)$ . For  $v \in V(G)$ , define its  $i$ th neighborhood as  $N_i(v) = \{u \in V(G) : d(u, v) = i\}$ . The  $r$ th power of a graph  $G$ , denoted  $G^r$ , is constructed from  $G$  by adding an edge between two vertices  $x$  and  $y$  when they are within distance  $r$  in  $G$ . Define the diameter of  $G$ ,  $\text{diam}(G)$ , as the minimal  $r$  such that  $G^r$  is complete (alternatively, the maximal distance between two vertices). Denote the number of edges of  $G$  by  $e(G)$ . For  $v \in V(G)$  and a set of vertices  $S$ , define  $e^r(v, S) = |\{u \in S : d(v, u) \leq r\}|$ .

The Cayley graph of a subset  $A \subseteq \mathbb{Z}_p$  is constructed on the vertex set  $\mathbb{Z}_p$ . For two distinct vertices  $x, y \in \mathbb{Z}_p$ , we define  $xy$  to be an edge whenever  $x - y \in A$  or  $y - x \in A$ . The following is a consequence of the Cauchy-Davenport Theorem (usually stated in the language of additive number theory [1]).

**Theorem 1.** *Let  $p$  be a prime,  $A$  a subset of  $\mathbb{Z}_p$ , and  $G$  the Cayley graph of  $A$ . Then for any integer  $r < \text{diam}(G)$ :*

$$e(G^r) \geq r e(G).$$

If we take  $A$  to be the arithmetic progression  $\{a, 2a, \dots, ka\}$ , then equality holds in this theorem for all  $r < \text{diam}(G)$ . We might look for analogues of Theorem 1 for more general graphs  $G$ . In particular since these Cayley graphs are always regular and (when  $p$  is prime) connected, we might focus on regular, connected  $G$ . In [2] Hegarty proved the following theorem:

**Theorem 2.** *Suppose  $G$  is a regular, connected graph with  $\text{diam}(G) \geq 3$ . Then we have*

$$e(G^3) \geq (1 + \epsilon) e(G),$$

with  $\epsilon \approx 0.087$

In other words, the cube of  $G$  retains the original edges of  $G$  and gains a positive proportion of new ones. In Section 3 we prove this theorem with an improved constant of  $\epsilon = \frac{1}{6}$ . The requirement of regularity cannot be easily dropped, as shown in [2].

Theorem 2 leads to the question of how the growth behaves for other powers of the  $G$ . Note that Theorem 2 cannot be used recursively to obtain such a result – since the cube of a regular graph is not necessarily regular. In [2] it was shown that no equivalent of Theorem 2 exists with  $G^3$  replaced by  $G^2$ , and it was asked what happens for higher powers. In this note we address that question.

## 2 Main Result

We prove the following theorem:

**Theorem 3.** *Suppose  $G$  is a regular, connected graph, and  $r \leq \text{diam}(G)$ . Then we have:*

$$e(G^r) \geq \left( \left\lceil \frac{r}{3} \right\rceil - 1 \right) e(G).$$

*Proof.* Let the degree of each vertex be  $d$ . Fix some  $v$  with  $N_{\text{diam}(G)}(v)$  nonempty.

Consider any vertex  $u \in V(G)$ . Then for any  $j$  satisfying  $d(u, v) - r < j \leq d(u, v)$ , there is a  $w_j \in N_j(v)$  such that  $d(u, w_j) < r$ . For such a  $w_j$ , all vertices  $x \in N_1(w_j)$  have  $d(u, x) \leq r$ . All such  $x$  are contained in  $N_{j-1}(v) \cup N_j(v) \cup N_{j+1}(v)$ , hence

$$e^r(u, N_{j-1}(v) \cup N_j(v) \cup N_{j+1}(v)) \geq d. \quad (1)$$

Note that each  $j \in \{d(u, v) - 3, d(u, v) - 6, \dots, d(u, v) - 3 \left( \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil - 1 \right)\}$  satisfies  $d(u, v) - r < j \leq d(u, v)$ . Summing the bound (1) over all these  $j$ , noting that any edge is counted at most once, we obtain

$$e^r(u, N_0(v) \cup \dots \cup N_{d(u, v)-2}(v)) \geq \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil d - d.$$

Now we sum this over all  $u \in G$ . Note that since the edges counted above go from some  $N_i(v)$  to  $N_j(v)$  with  $j < i$ , each edge is counted at most once. Also we haven't yet counted any of the original edges of  $G$ , so we might as well add them. Hence

$$\begin{aligned} e(G^r) &\geq \sum_{u \in G} e^r(u, N_0(v) \cup \dots \cup N_{d(u, v)-2}(v)) + e(G) \\ &\geq \sum_{u \in G} \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil d - |V(G)|d + e(G) \\ &= \sum_{u \in G} \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil - e(G). \end{aligned} \quad (2)$$

Obviously there was nothing particularly special about  $v$ . We can get a similar expression using  $v' \in N_{\text{diam}(G)}(v)$ , namely

$$e(G^r) \geq \sum_{u \in G} \left\lceil \frac{1}{3} \min\{d(u, v'), r\} \right\rceil - e(G). \quad (3)$$

Averaging (2) and (3) we get

$$e(G^r) \geq \frac{1}{2} \sum_{u \in G} \left( \left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil + \left\lceil \frac{1}{3} \min\{d(u, v'), r\} \right\rceil \right) d - e(G). \quad (4)$$

Note that for any  $u \in V(G)$  we have

$$\left\lceil \frac{1}{3} \min\{d(u, v), r\} \right\rceil + \left\lceil \frac{1}{3} \min\{d(u, v'), r\} \right\rceil \geq \left\lceil \frac{r}{3} \right\rceil. \quad (5)$$

This is because  $d(u, v) + d(u, v') \geq d(v, v') = \text{diam}(G) \geq r$ . Putting the bound (5) into the sum (4) we obtain

$$e(G^r) \geq \frac{|V(G)|d}{2} \left\lceil \frac{r}{3} \right\rceil - e(G) = \left\lceil \frac{r}{3} \right\rceil e(G) - e(G).$$

Thus the theorem is proven.  $\square$

### 3 Cubes

Note that for  $r \leq 6$  the bounds in Theorem 3 are trivial. In particular it says nothing about the increase in the number of edges of the cube of a regular, connected graph. Such an increase was already demonstrated by Hegarty in Theorem 2. Here we give an alternative proof of that theorem, yielding a slightly better constant.

**Theorem 4.** *Suppose  $G$  is a regular, connected graph with  $\text{diam}(G) \geq 3$ . Then we have*

$$e(G^3) \geq \left(1 + \frac{1}{6}\right) e(G).$$

*Proof.* Let the degree of each vertex be  $d$ . Note that as  $G$  is regular, and not complete, every  $v \in V(G)$  will have a non-neighbour in  $G$ . Together with connectedness this implies that each  $v \in V(G)$  has at least one new neighbour in  $G^2$ . This implies the theorem for  $d \leq 6$ . For the remainder of the proof, we assume that  $d > 6$ . The proof rests on the following colouring of the edges of  $G$ : For an edge  $uv$  in  $G$ , colour

$$\begin{aligned} uv & \text{ \textbf{red} if } |N_1(u) \cap N_1(v)| > \frac{2}{3}d, \\ uv & \text{ \textbf{blue} if } |N_1(u) \cap N_1(v)| \leq \frac{2}{3}d. \end{aligned}$$

Notice that if  $uv$  is a blue edge, then there are at least  $\frac{4}{3}d - 1$  neighbours of  $u$  in  $G^2$ . This is because  $u$  will be connected to everything in  $N_1(u) \cup N_1(v)$  except itself, and  $|N_1(u) \cup N_1(v)| \geq \frac{4}{3}d$  for  $uv$  blue. If, in addition, we have some  $x$  connected to  $u$  by an edge (of any colour), then  $x$  will be at distance at most 3 from everything in  $N_1(u) \cup N_1(v) \setminus \{x\}$ . Hence  $x$  will have at least  $\frac{4}{3}d - 1$  neighbours in  $G^3$ .

Partition the vertices of  $G$  as follows:

$$\begin{aligned} B &= \{v \in V(G) : v \text{ has a blue edge coming out of it}\}, \\ R &= \{v \in V(G) : v \notin B \text{ and there is a } u \in B \text{ such that } uv \text{ is an edge}\}, \\ S &= V(G) \setminus (B \cup R). \end{aligned}$$

By the above argument, if  $v$  is in  $B \cup R$ , then  $e^3(v, V(G)) \geq \frac{4}{3}d - 1$ . Recall that each  $u \in S$

will have at least one new neighbour in  $G^2$ , giving  $e^3(u, V(G)) \geq d + 1$ . Summing these two bounds over all vertices in  $G$ , noting that any edge is counted twice, gives

$$\begin{aligned}
2e(G^3) &\geq \left(\frac{1}{3}d - 1\right) |B \cup R| + (d + 1)|S| \\
&= \left(\frac{4}{3}d - 1\right) |B \cup R| + (d + 1)(|V(G)| - |B \cup R|) \\
&= \frac{7}{6}d|V(G)| + \frac{1}{3} \left(|B \cup R| - \frac{1}{2}|V(G)|\right) (d - 6) \\
&= \frac{7}{3}e(G) + \frac{1}{3} \left(|B \cup R| - \frac{1}{2}|V(G)|\right) (d - 6).
\end{aligned}$$

Recall that we are considering the case when  $d > 6$ . Thus to prove that  $e(G^3) \geq \frac{7}{6}e(G)$ , it suffices to show that  $|B \cup R| \geq \frac{1}{2}|V(G)|$ . To this end we shall demonstrate that  $|S| \leq |R|$ . First however we need a proposition helping us to find blue edges in  $G$ .

**Proposition 5.** *For any  $v \in V(G)$  there is some  $b \in B$  such that  $d(v, b) \leq 2$ .*

*Proof.* Suppose  $d(v, u) = 3$ . Then there are vertices  $x$  and  $y$  such that  $\{v, x, y, u\}$  forms a path between  $u$  and  $v$ . We will show that one of the edges  $vx$ ,  $xy$  or  $yu$  is blue. This will prove the proposition assuming that there are *any* blue edges to begin with. However, it also shows the existence of blue edges because  $\text{diam}(G) \geq 3$ .

So, suppose that the edges  $vx$  and  $uy$  are red. Then we have  $|N_1(v) \cap N_1(x)| > \frac{2}{3}d$ , and  $|N_1(u) \cap N_1(y)| > \frac{2}{3}d$ . Using this and  $|N_1(u) \cap N_1(v)| = \emptyset$  gives

$$\begin{aligned}
|N_1(x) \cup N_1(y)| &\geq |(N_1(x) \cup N_1(y)) \cap N_1(v)| + |(N_1(x) \cup N_1(y)) \cap N_1(u)| \\
&\geq |N_1(x) \cap N_1(v)| + |N_1(y) \cap N_1(u)| \\
&> \frac{4}{3}d.
\end{aligned}$$

Therefore  $|N_1(x) \cap N_1(y)| = 2d - |N_1(x) \cup N_1(y)| \leq \frac{2}{3}d$ . Hence  $xy$  is blue, proving the proposition.  $\square$

Now we will show that  $|S| \leq |R|$ . Suppose  $r \in R$ . By the definition of  $R$ , there is a  $b \in B$  such that  $rb$  is an edge. This edge is necessarily red as  $r \notin B$ . Using  $N_1(b) \subseteq B \cup R$ , we have  $|N_1(r) \cap (B \cup R)| \geq |N_1(r) \cap N_1(b)| > \frac{2}{3}d$ . Hence

$$|N_1(r) \cap S| \leq \frac{1}{3}d. \quad (6)$$

Suppose  $s \in S$ . Proposition 5 implies that there is some  $r \in R$  such that  $sr$  is an edge. Since  $sr$  is red, we have  $|N_1(s) \cap N_1(r)| > \frac{2}{3}d$ . Using this, the fact that  $N_1(s) \subseteq R \cup S$ , and (6), gives

$$\begin{aligned}
|N_1(s) \cap R| &\geq |N_1(s) \cap N_1(r) \cap R| \\
&= |N_1(s) \cap N_1(r)| - |N_1(s) \cap N_1(r) \cap S| \\
&\geq |N_1(s) \cap N_1(r)| - |N_1(r) \cap S| \\
&> \frac{1}{3}d.
\end{aligned} \quad (7)$$

Double-counting the edges between  $S$  and  $R$  using the bounds (6) and (7) gives a contradiction unless  $|S| \leq |R|$ . Therefore  $|B \cup R| \geq \frac{1}{2}|V(G)|$  as required.  $\square$

## 4 Discussion

Theorem 3 answers the question of giving a lower bound on the number of edges that are gained by taking higher powers of a graph. We obtain growth that is linear with  $r$  – just as in Theorem 1.

- The constant  $\lceil \frac{1}{3}r \rceil$  in Theorem 3 cannot be improved to something of the form  $\lambda r$  with  $\lambda > \frac{1}{3}$ . To see and consider the following sequence of graphs  $H_r(d)$  as  $d$  tends to infinity: Take disjoint sets of vertices  $N_0, \dots, N_r$ , with  $|N_i| = d - 1$  if  $i \equiv 0 \pmod{3}$  and  $|N_i| = 2$  otherwise. Add all the edges within each set and also between neighboring ones. So if  $u \in N_i$ ,  $v \in N_j$ , then  $uv$  is an edge whenever  $|i - j| \leq 1$  (see Figure 1).

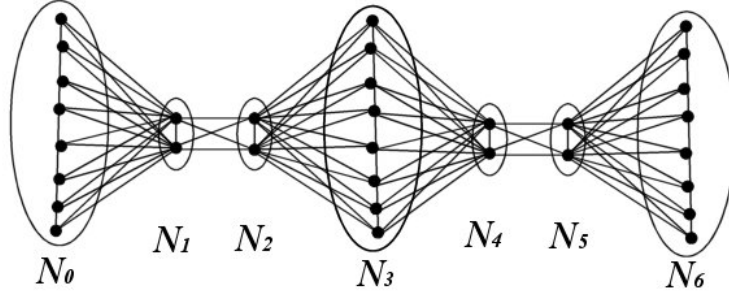


Figure 1: The graph  $H_6(9)$ .

The number of edges in  $H_r(d)$  is at least the number of edges in the larger classes which is  $\lceil \frac{1}{3}(r+1) \rceil \binom{d-1}{2}$ .

The  $r$ th power  $H_r(d)^r$  has less than  $\binom{|V(G)|}{2}$  edges which is less than  $\binom{\lceil \frac{1}{3}(r+1) \rceil (d+3)}{2}$ . Therefore,

$$\limsup_{d \rightarrow \infty} \frac{e(H_r(d)^r)}{e(H_r(d))} \leq \lim_{d \rightarrow \infty} \frac{\binom{\lceil \frac{1}{3}(r+1) \rceil (d+3)}{2}}{\lceil \frac{1}{3}(r+1) \rceil \binom{d-1}{2}} = \left\lceil \frac{1}{3}(r+1) \right\rceil.$$

The graphs  $H_r(d)$  are not regular, but if  $r \not\equiv 2 \pmod{3}$ , it is possible to remove a small (less than  $|V(G)|$ ) number of edges from the graphs and make them  $d$ -regular without losing connectedness (any cycle passing through all the vertices in  $N_1 \cup \dots \cup N_{r-1}$  would work). Call these new graphs  $\hat{H}_r(d)$ . By the same argument as before we have

$$\limsup_{d \rightarrow \infty} \frac{e(\hat{H}_r(d)^r)}{e(\hat{H}_r(d))} \leq \left\lceil \frac{1}{3}(r+1) \right\rceil.$$

If  $r \equiv 2 \pmod{3}$ , a similar trick can be performed, but we'd need to start with  $|N_i| = d - 1$  if  $i \equiv 1 \pmod{3}$  and  $|N_i| = 2$  otherwise.

So the factor of  $\frac{1}{3}$  cannot be improved for regular graphs. All these examples are inspired by one given in [2] to show that for any  $\epsilon$  there are regular graphs  $G$  with  $e(G^2) < (1 + \epsilon)e(G)$ .

- Despite the above example, there is certainly room for further improvement in Theorems 3 and 4. In particular, Theorem 4 doesn't seem tight in any way. The graphs  $\hat{H}_r(d)$  seem to give essentially the slowest possible growth for *all* powers of regular graphs. Considering the graphs  $H_3(d)$  leads to the conjecture of

$$e(G^3) \geq 2e(G),$$

for  $G$  regular, connected, and  $\text{diam}(G) \geq 3$ .

A shortcoming of Theorem 3 is that it only gives a good bound if the diameter of  $G$  is close to  $r$ . When this is not the case, the number of edges in  $G^r$  seems to grow faster. It would be interesting to obtain a good lower bound on  $e(G^r)$  involving both  $r$  and  $\text{diam}(G)$ .

- All the questions from this paper and [2] could be asked for directed graphs. In particular one can define directed Cayley graphs for a set  $A \subseteq \mathbb{Z}_p$  by letting  $xy$  be a directed edge whenever  $x - y \in A$ . Then the Cauchy-Davenport Theorem implies an identical version of Theorem 1 for directed Cayley graphs. In this setting it is easy to show that there is growth even for the square of an out-regular oriented graph  $D$  (a directed graph where for a pair of vertices  $u$  and  $v$ ,  $uv$  and  $vu$  are not both edges). In particular, we have

$$e(D^2) \geq \frac{3}{2} e(D). \quad (8)$$

This occurs because every vertex  $v$  has  $|N_2^{\text{out}}(v)| \geq \frac{1}{2}|N_1^{\text{out}}(v)|$  in an out-regular oriented graph. It's easy to see that this is best possible for such graphs. One can construct out-regular oriented graphs with an arbitrarily large proportion of vertices  $v$  satisfying  $|N_2^{\text{out}}(v)| = \frac{1}{2}|N_1^{\text{out}}(v)|$ .

However if we insist on *both* in and out-degrees to be constant, (8) no longer seems tight. Such graphs are always Eulerian. In [3] there is a conjecture attributed to Jackson and Seymour that if an oriented graph  $D$  is Eulerian, then  $e(D^2) \geq 2e(D)$  holds. If this conjecture were proved, it would be an actual generalization of the directed version of Theorem 1, as opposed to the mere analogues proved above.

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## References

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